An Image Curvature Microscope

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Séminaire Jean Serra - 70 ans
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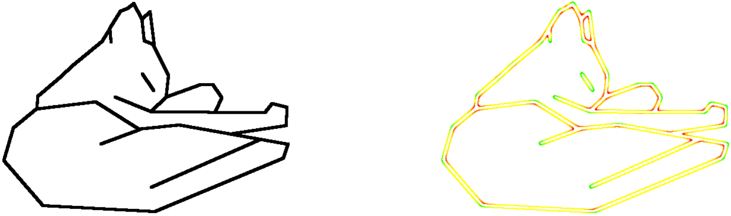
Overview

Role of Curvature in Visual Perception



(a). Attneave's cat (b). Curvature map.

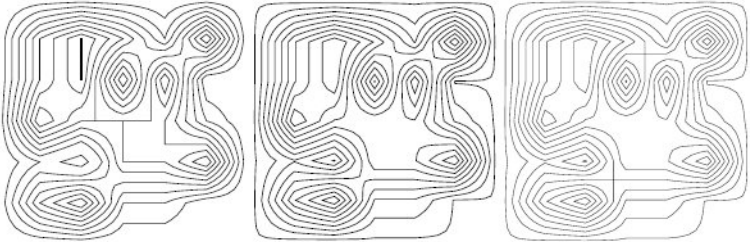
Figure: *"Information is concentrated along contours (i.e., regions where color changes abruptly), and is further concentrated at those points on a contour at which its direction changes most rapidly (i.e., at angles or peaks of curvature)" - Attneave).*

Image interpolations: Whittaker - Shannon

Digital images can be modeled as

- zero order interpolations: piecewise constant functions (*block interpolation*);
- first order interpolations: continuous functions, taking the given values at the centers of the pixels and being affine on the corresponding edges (*bilinear interpolation*);
- higher order spline interpolations (e.g. bicubic).

Bilinear tree of level lines



Bilinear interpolation in a dual pixel can locally be written as

$$u(x, y) = axy + bx + xy + d$$

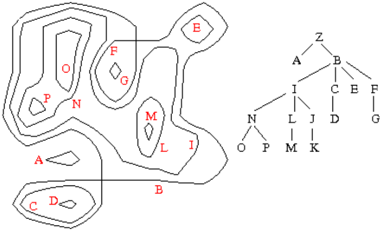
where the parameters a, b, c, d are given by the values taken at four adjacent pixels. Level lines are then concatenations of pieces of hyperbolae and straight lines.

Bilinear tree of level lines

- One can decompose an image into its level lines at quantized levels.

$$\mathcal{T} = \{\Sigma^{\lambda,i}\}_{\lambda\in\Lambda, i\in F_\lambda};$$

- The set is ordered in a tree structure, induced by the geometrical inclusion.



Algorithm (Monasse Guichard '98)

A fast algorithm, the Fast Level Set Transform (FLST) performs the decomposition of an image into a tree of shapes (subsequently, in a tree of level lines).

Image reconstruction

Algorithm (Ciomaga, Monasse, Morel, '09)

Construct an image from its topographic map

- *walk the tree in pre-order (parent before children)*
- *fill the interior of the current level line* $\Sigma = \{P_k(x_k, y_k)\}_{1 \leq k \leq N}$ *with its level* λ*:*
  - *find intersections of the boundary with all horizontal lines of equation* $y = i$ *and write the abscissas in an ordered set*
  - *a pixel* (j, i) *is inside the polygon if and only if* j *is within an interval* $[x^i_{2k+1}, x^i_{2k+2}]$.

Image Reconstruction

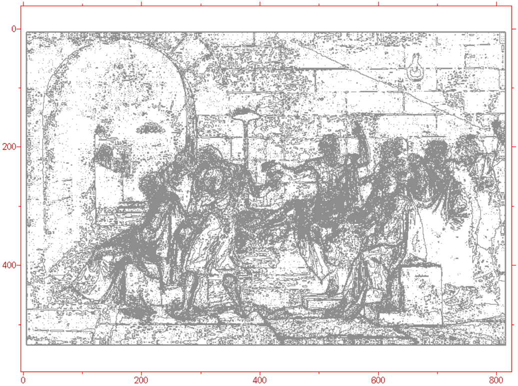


Figure: Level lines at quantized levels, quantization step $s = 16$.

Image Reconstruction

Figure: Image reconstructed from the whole tree of level lines, at half-integer gray values.

Curvatures in digital images

The scalar curvature of a C^2 image at a nonsingular point $\mathbf{x_0}$ is defined by

$$\operatorname{curv}(u)(\mathbf{x_0}) = \frac{u_{xx}u_y^2 - 2u_{xy}u_xu_y + u_{yy}u_x^2}{(u_x^2 + u_y^2)^{3/2}}(\mathbf{x_0}). \tag{1}$$

This scalar curvature at $\mathbf{x_0}$ is linked to the vectorial curvature $\kappa(\mathbf{x_0})$ of the level line passing by $\mathbf{x_0}$ via

$$\kappa(\mathbf{x_0}) = -\operatorname{curv}(u)(\mathbf{x_0}).\frac{Du}{|Du|}(\mathbf{x_0}). \tag{2}$$

Thus, curvatures in digital images can be computed in two quite different ways.

Multiscale curvature

A previous smoothing is necessary, which introduces a new parameter, the smoothing *scale*. Hence the notion of *curvature scale space* which will be associated with curve or image evolutions.

Problem

Smoothing algorithms in the computer vision literature deal with either

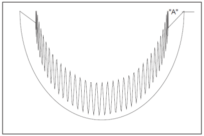
- *level lines: curve/affine shortenings*
- *level sets: threshold dynamics*
- *or with the image itself: FDSs and stack filters*

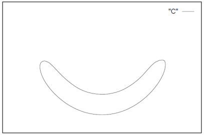
Curvature Flows

- Data: closed curve Γ_0
- Perform curvature driven flows

$$\Sigma_0 \mapsto \Sigma_t$$

$$\frac{\partial x}{\partial t} = |k|^{\sigma - 1} k \overrightarrow{n}$$





Questions

- *well posedness; existence and regularity of solutions ;*
- *numerical approximation schemes;*
- *morphological compatiility: inclusion principle .*

Local heat equation: $x_t(t, s) = x_{ss}(t, s)$

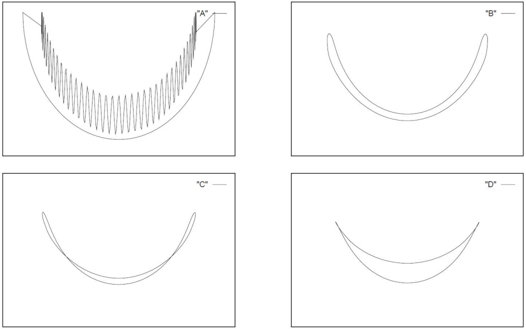


Figure: Curve evolution by the heat equation .The evolving curve can, however, develop self-crossings (as in C) or singularities (as in D).

Dynamic curve evolution: $x_t(t, s) = \mathbf{k}(t, s)$

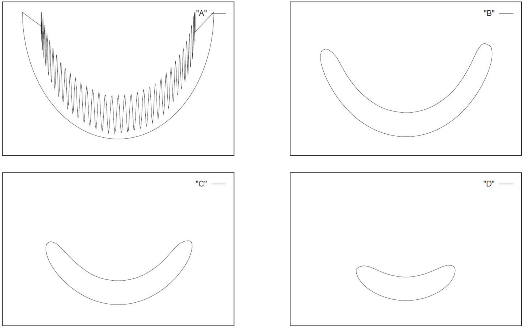


Figure: Curve evolution by the renormalized heat equation. The evolved curve is smooth for all times, eventually becomes convex and shrinks to a point.

Dynamic curve evolution: nonlocal heat equation

Algorithm (Mackworth Mockhtarian '92)

- *Convolve the curve* x_n*, parameterized by its length parameter* $s_n \in [0, L_n]$*, with a Gaussian* G_h*, where* h *is small.*

$$x_{n+1}(s_n) = G_h * x_n(s_n).$$

- *Reparametrize* x_{n+1} *by its length parameter* $s_{n+1} \in [0, L_{n+1}]$*.*

Level set methods

Algorithm (Stack filter and threshold dynamics)

- *Decompose u_0 in its upper level sets and consider the characteristic function $\chi_\lambda(\cdot)$ of each upper level set $X_\lambda u_0$;*

$$u_0 \mapsto \{X_\lambda u_0\}_\lambda.$$

- *Solve mean curvature motion for $\chi_\lambda(\cdot)$ until the scale t.*

$$\psi_\lambda(t,\cdot) = FDS(\chi(\cdot))(t).$$

- *Get back the image by thresholding*

$$u(t,x) = \lambda, \forall x \ s.t. \ \psi_\lambda(x) \geq 1/2.$$

Level set methods

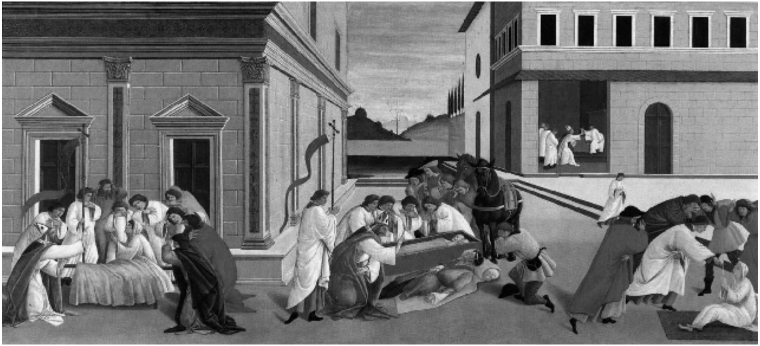


Figure: Level set method (BMO algorithm) for mean curvature evolution, at renormalized scale $l = 2$.

Level set methods

Figure: Level set method (BMO algorithm) for mean curvature evolution, at renormalized scale $l = 2$.

Level Lines Shortening

Subpixel algorithm based on the topological structure of the level lines

$$
\begin{tikzcd}
u_0(\cdot) \arrow[r, "level\ \ lines\ \ extraction"] \arrow[d, dotted, "MCM/ACM"'] & \{\Sigma_0^{\lambda,i}\}_{\lambda,i} \arrow[d, "CS/AS"] \\
u(\cdot, t) & \{\Sigma_t^{\lambda,i}\}_{\lambda,i} \arrow[l, "reconstruction"']
\end{tikzcd}
$$

The scheme is monotonous and therefore ensures level lines order preserving.

Level lines Shortening

Algorithm (Ciomaga, Monasse, Morel '09)

Perform the LLS evolution of u_0 at scale t:

- *Extract the tree of level lines* $\{\Sigma_0^{\lambda,i}\}_{\{i \in F_\lambda, \lambda\}}$;
- *Smooth each level line separately*

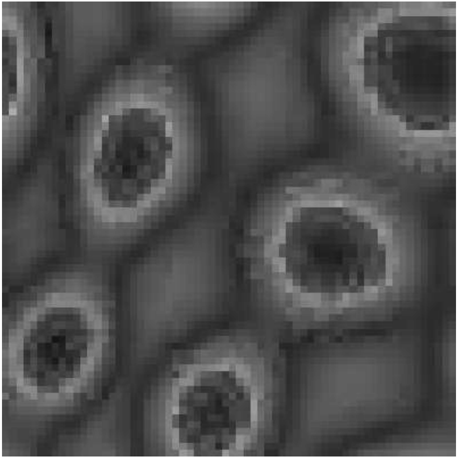
$$\Sigma_t^{\lambda,i} = \textit{ Curve Shortening Flow } (\Sigma_0^{\lambda,i})$$

- *Reconstruct the image by filling the interior laminas bounded by each level line* $\Sigma_t^{\lambda,i}$;

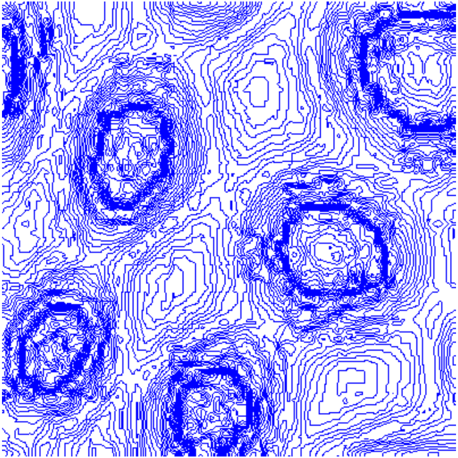
Algorithm (Moisan, Koepfler,'99)

Outline of this numerical chain in "Geometric Multiscale Representation of Numerical Images"

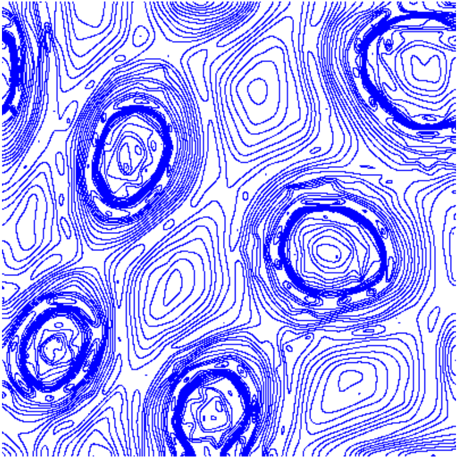
Level Lines Shortening (LLS)



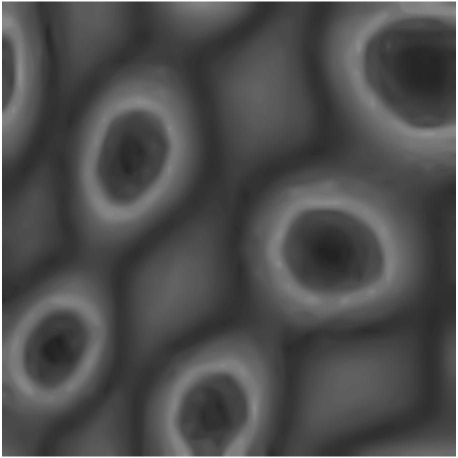
Level Lines Shortening (LLS)



Level Lines Shortening (LLS)



Level Lines Shortening (LLS)



Level lines Shortening

Theorem (Ciomaga, Morel '10)

Let $u_0 \in Lip(\Omega)$. Then $u(x,t) : \Omega \times [0,\infty) \to \mathbb{R}$ defined by the Level Lines Shortening evolution of u_0

$$u(x,t) = LLS(t)u_0(x), \forall x \in \mathbb{R}^2, \forall t \in [0,\infty)$$

is a viscosity solution for the mean curvature PDE, with the initial data u_0:

$$\begin{cases} u_t = (\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2}) u_{x_i x_j}, & \text{in } \mathbb{R}^2 \times [0,\infty) \\ u(\cdot, 0) = u_0, & \text{on } \mathbb{R}^2. \end{cases} \quad (3)$$

Local comparison principle and regularity

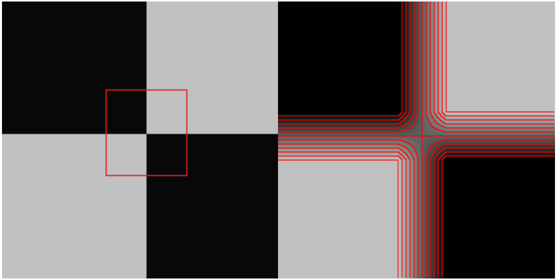


Figure: Original image and its level lines

Local comparison principle and regularity

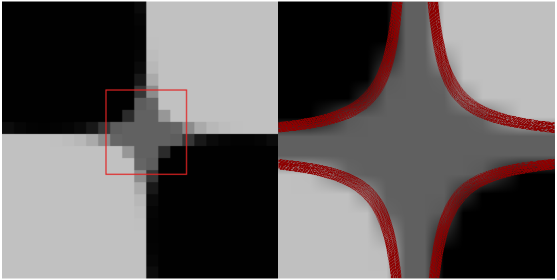


Figure: Level Lines Shortening (LLS)

Local comparison principle and regularity

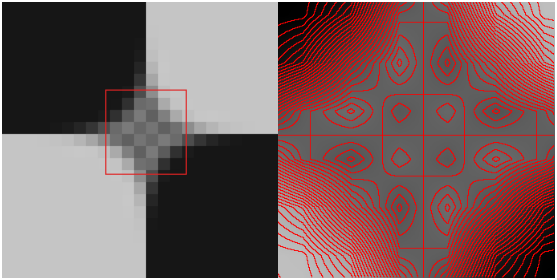


Figure: Finite difference scheme (FDS)

Local comparison principle and regularity

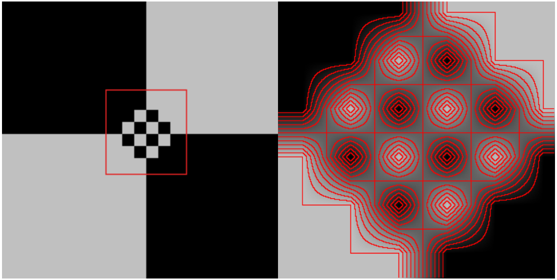


Figure: Stack filter, threshold dynamics applied to the FDS

Curvatures computed directly on level lines

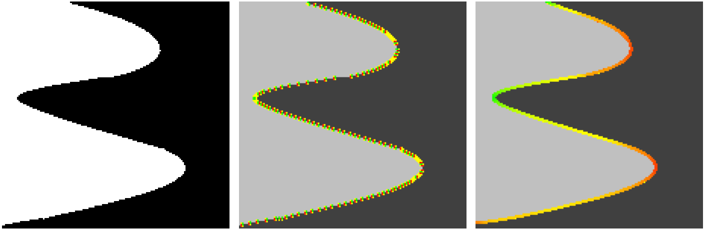


Figure: The curvature color display rule. Initial image, FDS and LLS.

Discrete curvature for a polygonal line.

We recall that each level line is stored as a set of ordered points

$$\Sigma = \{P_i(x_i, y_i)\}_{i=0..n}, \text{ with } P_0 = P_n.$$

The discrete scalar curvature k_i computed at each vertex P_i is obtained as the inverse of the circumscribed radius R_i of the triangle $P_{i-1}P_iP_{i+1}$.

Lemma

The curvature at vertex P_i is given by

$$k_i = 2\frac{u_i^1 u_{i+1}^2 - u_i^2 u_{i+1}^1}{u_i u_{i+1} v_i}. \tag{4}$$

Subpixel curvature algorithm

Algorithm (Ciomaga, Monasse, Morel, '10)

Compute the image curvature microscope

- *Extract the tree of level lines* $\{\Sigma_0^{\lambda,i}\}_{\{i \in F_\lambda, \lambda\}}$;
- *Perform uniform, fine sampling uniformly each level line;*
- *Smooth each level line separately*

$$\Sigma_t^{\lambda,i} = \textit{ Curve Shortening Flow } (\Sigma_0^{\lambda,i})$$

- *Compute the discrete curvatures at each vertex;*
- *Register at each dual pixel the average of all discrete curvatures computed in and create thus the curvature image.*

Signed and topological curvatures

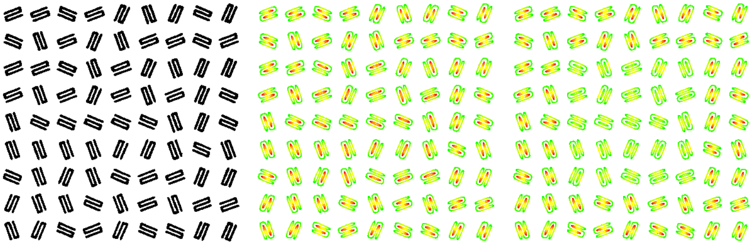


Figure: Original image, signed curvatures and topological curvatures

Curvature Microscope

Figure: Original image, 2X zoom and 4X zoom of the up-right corner. A zoom is necessary to observe the single curvatures.

Curvature Microscope

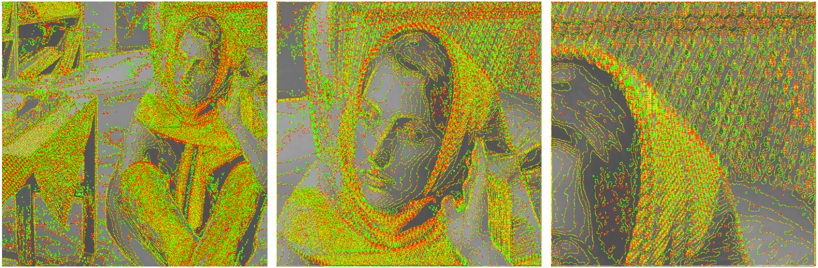


Figure: Curvature map computed on the original level lines with a quantization step $s = 16$.

Curvature Microscope

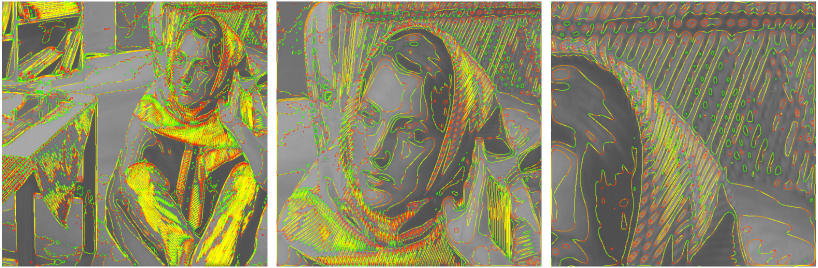


Figure: Curvature map computed on shortened level lines at normalized scales $l = 1$, $l = 2$, and $l = 4$.

A closer look at Attneave's cat

Figure: Zoom on the Attneave cat, its corresponding level lines and curvatures.

A closer look at Attneave's cat

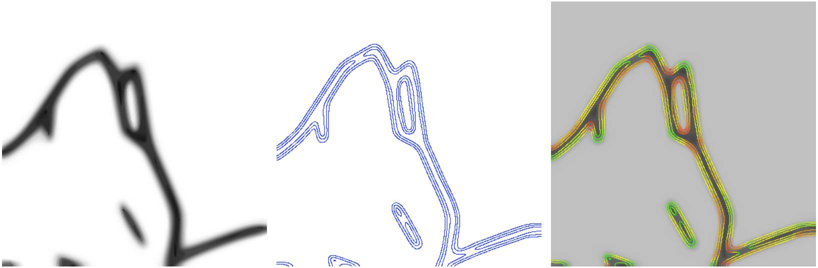


Figure: LLAS evolution, smoothed level lines and filtered curvature map

Graphics and aliasing

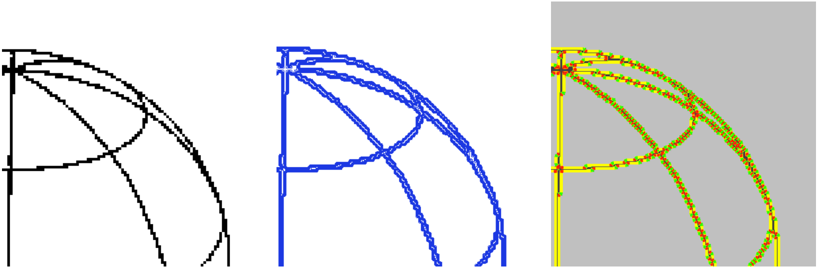


Figure: Original image, its corresponding level lines and curvatures.

Graphics and aliasing

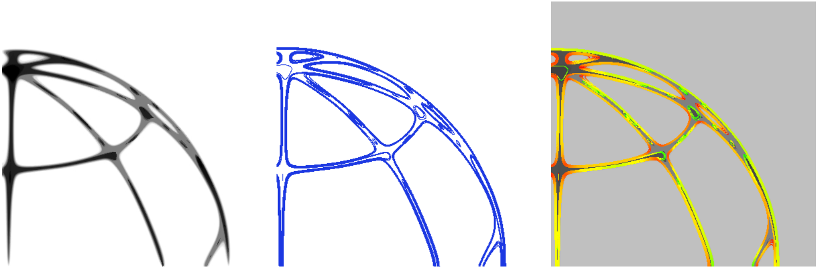


Figure: LLAS evolution, affine smoothed level lines and curvature map after filtering.

Bacteria morphologies

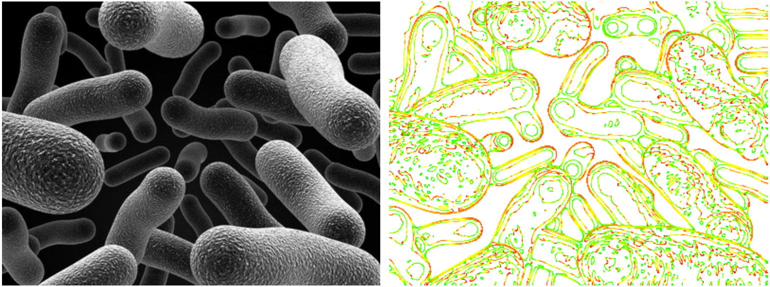


Figure: Original bacteria image and the corresponding curvature map.

Digital elevation models

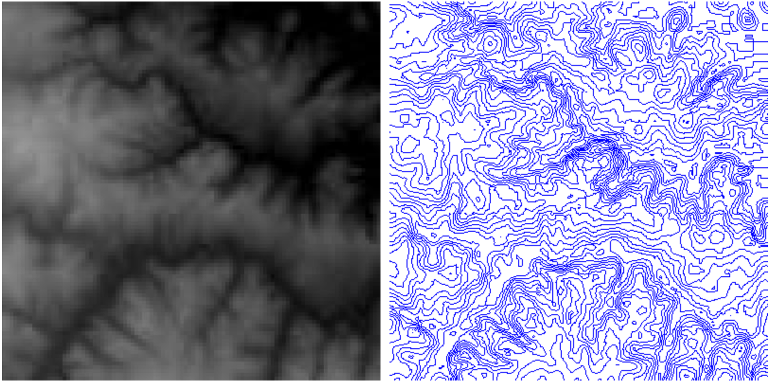


Figure: Digital elevation map and its corresponding level lines.

Digital elevation models

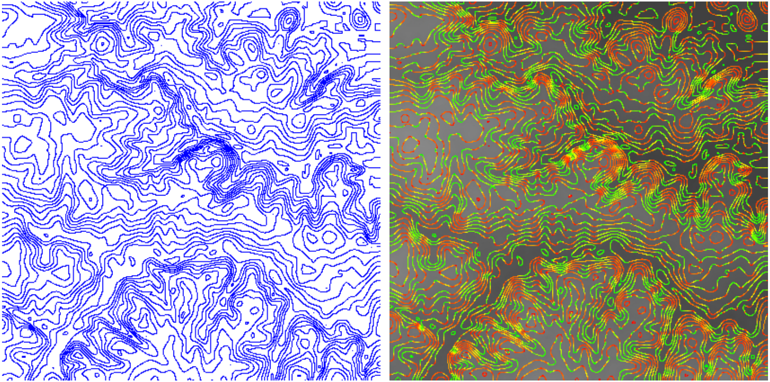


Figure: The affine smoothed level lines and their curvature map.

JPEG artefact reduction

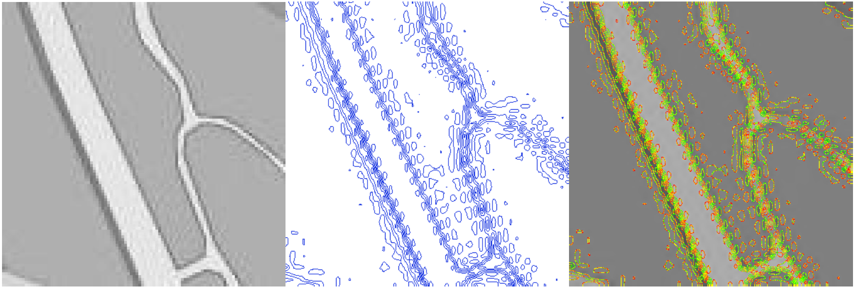


Figure: Piece of map with roads, its corresponding level lines and curvatures.

JPEG artefact reduction

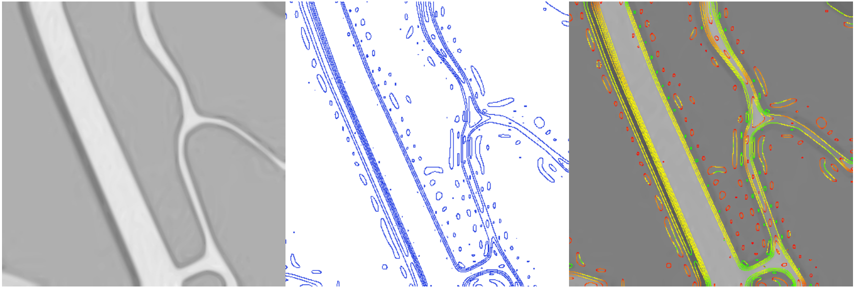


Figure: LLAS evolution, affine smoothed level lines and curvature map after filtering.

Paitings *sfumato technique*

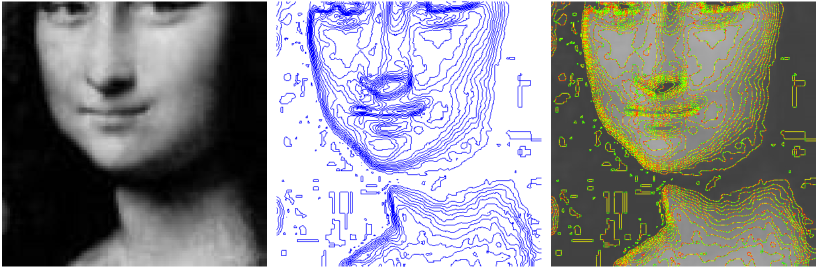


Figure: Extraction with zoom of *Mona Lisa* photograph, its corresponding level lines and curvatures.

Paitings *sfumato technique*

Figure: LLAS evolution, affine smoothed level lines and curvature map after filtering.

Text processing

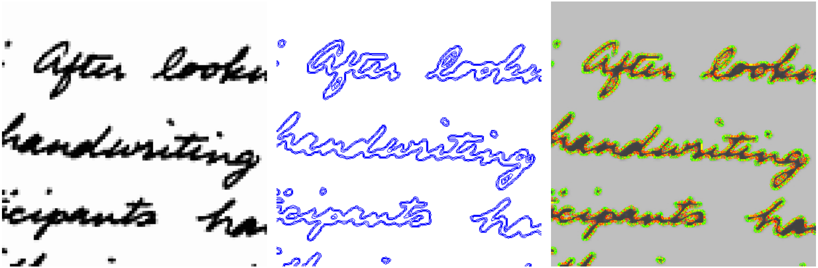


Figure: Original handwriting, corresponding level lines and curvatures.

Text processing

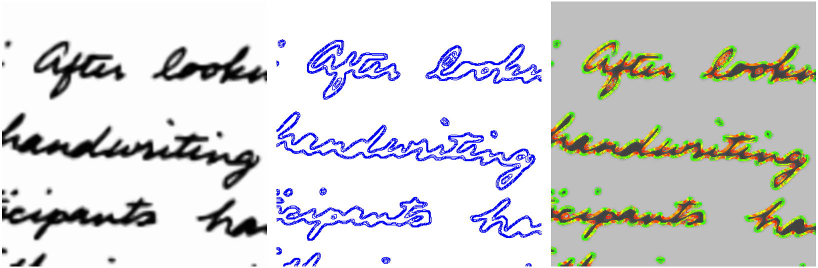


Figure: LLAS evolution, affine smoothed level lines and curvature map after filtering.

Fingerprints restoration and discrimination

Figure: Original fingerprint, Level Lines Affine Shortening and its Curvature map.

Conclusion

- *The first outcome of the Level lines Shortening algorithm is the evolved image, which presents some sort of denoising, simplification, and desaliasing;*
- *The main outcome is an accurate curvature estimate on all level lines;*
- *A powerful visualization tool, due to the fact that all level lines are polygons with real coordinates allows to zoom in the image at an arbitrary resolution;*

Conclusion

It runs online at

http://www.ipol.im