



Roussos-Maragos Tensor-Driven Diffusion for Image Interpolation

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Overview

Roussos and Maragos proposed a method for image interpolation in "Reversible interpolation of vectorial images by an anisotropic diffusion-projection PDE" [9]. An earlier version was also published in conference paper [8].

Given a discretely sampled image v , the method finds an image u such that

$$v_n = (h * u)(n), \quad \text{for all } n \in \mathbb{Z}^2.$$

where h is the (assumed known) point spread function and $*$ denotes convolution.

The method is inspired by tensor-driven diffusion works of Tschumperlé [6], [7] and Weickert [3]. Roussos and Maragos propose interpolation by evolving a diffusion equation to steady state,

$$\partial_t u = P_0(\text{div}(T\nabla u))$$


where T is a tensor determined from image structure tensor and the diffusion is orthogonally projected to agree with the observed data. This diffusion is based on the general anisotropic diffusion model proposed by Weickert [3]. The method can be applied to grayscale, color, or general vector-valued images.

Content

- Overview
- References
- Online Demo
- Initial Interpolation
- Structure Tensor
- Tensor-Driven Diffusion
- Projection onto the Solution Set
- Algorithm
- Implementation
- Examples

References

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Online Demo

An [online demo](#) of this algorithm is available.

Initial Interpolation

First, an initial interpolation u_0 is computed by Fourier zero-padding with deconvolution,

$$\hat{u}_0(\xi) = \begin{cases} \frac{1}{\hat{h}(\xi)} \sum_{n \in \mathbb{Z}^2} v_n e^{-i2\pi n \xi}, & \text{if } \xi \in [-\frac{1}{2}, +\frac{1}{2}] \times [-\frac{1}{2}, +\frac{1}{2}], \\ 0 & \text{otherwise,} \end{cases}$$

where \hat{h} is the two-dimensional Fourier transform of the point spread function h ,

$$\hat{h}(\xi) := \int_{\mathbb{R}^2} h(x) e^{-i2\pi(x \cdot \xi)} dx.$$

For the division, it is assumed that $\hat{h}(\xi) \neq 0$ in $[-\frac{1}{2}, +\frac{1}{2}] \times [-\frac{1}{2}, +\frac{1}{2}]$.

This interpolation is used as an initialization for the tensor-driven diffusion because it satisfies $v_n = (h * u_0)(n)$ for all n . It is well-known that Fourier interpolation produces significant ringing artifacts, so the goal of the diffusion is to remove the ringing.

Structure Tensor

As introduced by Bigün and Granlund [1] and Förstner and Gülch [2], the image structure tensor is

$$J(\nabla u) := \begin{pmatrix} \partial_{x_1} u \\ \partial_{x_2} u \end{pmatrix} \begin{pmatrix} \partial_{x_1} u & \partial_{x_2} u \end{pmatrix}.$$

At each point in the image, $J(\nabla u)$ is a 2×2 symmetric matrix. Roussos and Maragos use the smoothed structure tensor,

$$J_\rho(\nabla u_\sigma) = G_\rho * J(\nabla(G_\sigma * u)),$$

where G_σ and G_ρ are Gaussians with standard deviations σ and ρ , which control the amount of pre- and post-smoothing. The post-smoothing convolution with G_ρ is applied separately to each component of the tensor. The image gradient ∇u can be discretized using centered differences. Alternatively, the gradient may be incorporated into the pre-smoothing convolution as

$$\nabla(G_\sigma * u) = (\nabla G_\sigma) * u,$$

and then approximated with discrete convolutions.

For a color image, the smoothed structure tensor is computed as the sum of the smoothed structure tensors for each channel.

Next, at every point in the image, the eigenvectors and eigenvalues of the the 2×2 matrix $J_\rho(\nabla u_\sigma)$ are computed. Since the matrices are guaranteed to be symmetric and real, the eigenvalues are real and the eigenvectors are real and orthogonal. Following Tschumperlé [6], [7], Roussos and Maragos construct the tensor T according to this eigensystem,

$$T(J_\rho(\nabla u_\sigma)) = \left(1 + \frac{1}{K^2}(\lambda_1 + \lambda_2)\right)^{-1/2} J\left(\frac{w_1}{|w_1|}\right) + \left(1 + \frac{1}{K^2}(\lambda_1 + \lambda_2)\right)^{-1} J\left(\frac{w_2}{|w_2|}\right),$$

where K is a parameter, $\lambda_1 \leq \lambda_2$ are the eigenvalues and w_1 and w_2 are the corresponding eigenvectors. The tensor is recomputed every n timesteps of the diffusion.

To compute the eigensystem of a matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, the eigenvalues are

$$\lambda_1 = \frac{1}{2}(a + c) - \sqrt{\frac{1}{4}(a + c)^2 - (ac - b^2)},$$

$$\lambda_2 = \frac{1}{2}(a + c) + \sqrt{\frac{1}{4}(a + c)^2 - (ac - b^2)},$$

and provided $b \neq 0$, the eigenvector corresponding to λ_1 is

$$w_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix}.$$

The other eigenvector w_2 is found as the orthogonal complement.

Tensor-Driven Diffusion

The interpolation is diffused according to the tensor as

$$\partial_t u = \operatorname{div}\left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} \nabla u\right),$$

where $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ denotes the components of the tensor field T from the previous section. To implement the diffusion, one approach is proposed in Weickert's book [3]. More recently, Weickert and Scharf [4] proposed a method that is faster yet also simpler: define the derivative approximations

$$F_x u = \frac{1}{32} \begin{pmatrix} -3 & 0 & 3 \\ -10 & 0 & 10 \\ -3 & 0 & 3 \end{pmatrix} * u, \quad F_y u = \frac{1}{32} \begin{pmatrix} 3 & 10 & 3 \\ 0 & 0 & 0 \\ -3 & -10 & -3 \end{pmatrix} * u,$$

then the diffusion is implemented explicitly using this approximation as

$$u^{\text{next}} = u + dt \left(F_x (a F_x u + b F_y u) + F_y (b F_x u + c F_y u) \right).$$

The filters F_x and F_y have optimal approximate rotation invariance among all 3×3 linear filters. Weickert and Scharf claim that rotation invariance is crucial for avoiding blur artifacts in the diffusion. Since F_x and F_y are 3×3 linear filters, the composition used in the formula above is effectively a 5×5

scheme.

Note that F_x and F_y are separable filters and have only two different filter coefficient values, so they can be implemented efficiently:

$$\begin{aligned}\tilde{u}_x(i, j) &= u(i+1, j) - u(i-1, j), \\ (F_x u)(i, j) &= \frac{10}{32}\tilde{u}_x(i, j) + \frac{3}{32}(\tilde{u}_x(i, j+1) + \tilde{u}_x(i, j-1)),\end{aligned}$$

and similarly for F_y .

For a color image, the diffusion scheme is computed independently to each channel. The channels are nevertheless coupled since they are all guided by the same tensor field, which is computed jointly over the channels.

Projection onto the Solution Set

The input is modeled as sampling the underlying continuous-domain image by $v = \text{sample}(h * u)$, so the solution is required to belong to the affine set

$$W_v = \{u : v(n) = (h * u)(n), \forall n \in \mathbb{Z}^2\}.$$

To impose this requirement, the diffusion is orthogonally projected onto W_v ,

$$\partial_t u = P_0(\text{div}(T\nabla u)).$$

where P_0 denotes orthogonal projection onto W_0 and the initial interpolation u_0 computed with Fourier zero-padding is in W_v . Then u is in W_v for all $t > 0$.

For numerical implementation, the solution is projected onto W_v after every n time steps and after the final time step. The projection is expensive to compute compared to the other steps of the algorithm, so it is helpful to reduce the number of projections that need to be computed.

We use the following normalization of the Fourier transform,

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-2\pi i x \cdot \xi} dx.$$

Before the main iteration, a function ϕ is precomputed,

$$\hat{\phi}(\xi) = \left(\sum_{k \in \mathbb{Z}^2} |\hat{h}(\xi - k)|^2 \right)^{-1/2} \hat{h}(\xi).$$

The projection onto W_v can then be implemented as

$$(\hat{P}_v(u))^\wedge(\xi) = \hat{u}(\xi) - \overline{\hat{\phi}(\xi)} \sum_{n \in \mathbb{Z}^2} \hat{\phi}(\xi - n) (\hat{u}(\xi - n) - \hat{u}_0(\xi - n)).$$

See derivation of P_v

Algorithm

Here we summarize the algorithm. First, the initial interpolation u_0 is computed by Fourier zero-padding with deconvolution and the projection function $\hat{\phi}$ is precomputed. The algorithm then iterates the main projection-diffusion loop:

1. Compute the tensor T .
2. Perform n explicit timesteps of $\partial_t u = \text{div}(T\nabla u)$ using the approximation

$$\text{div}\left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} \nabla u\right) \approx F_x(aF_x u + bF_y u) + F_y(bF_x u + cF_y u).$$

3. Orthogonally project the solution onto W_v with



$$\left(P_v(u)\right)^\wedge(\xi) = \hat{u}(\xi) - \overline{\hat{\varphi}(\xi)} \sum_{n \in \mathbb{Z}^2} \hat{\varphi}(\xi - n) (\hat{u}(\xi - n) - \hat{u}_0(\xi - n)).$$

The loop stops when either $\|u_{\text{cur}} - u_{\text{prev}}\|_2 \leq \text{tol}$ or when a maximum number of iterations N is reached.

For operations involving Fourier transforms, boundary artifacts are avoided by half-sample symmetric extension of the image. In the implementation, the image is extended by $(5 \times \text{scalefactor})$ pixels on each of the four borders of the image.

Implementation

This software is distributed under the terms of the simplified BSD license.

- source code [zip](#)  [tar.gz](#) 
- [online documentation](#)

Fourier transforms are implemented using the FFTW library. Please see the `readme.html` file or the online documentation for details.

Examples

The following examples demonstrate the method for factor-4 interpolation. The parameters used are

- point spread function h is a Gaussian with standard deviation 0.5 in units of input pixels
- $K = 1$
- $dt = 2$
- $\text{tol} = 0.1$
- $n = 5$ iterations
- $N = 50$ maximum iterations

In practice, the standard deviation of the Gaussian point spread function h must be tuned to approximate how the input image was sampled. The examples are shown using standard deviation 0.5, which provides moderate antialiasing. In the [online demo](#), the default value is 0.35, which is a reasonable model of the blurriness of typical images.

Input Image (86×79)

Tensor-Driven Diffusion, CPU time 5.216s



Input Image (86×79)



Tensor-Driven Diffusion, CPU time 2.149s

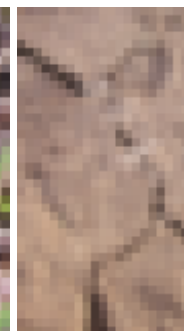
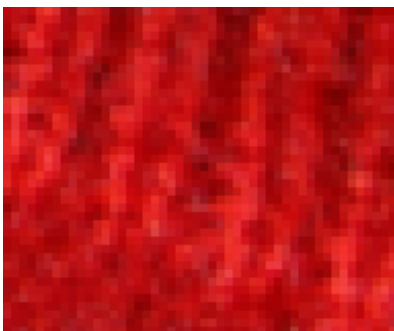


The next example demonstrates the method's good performance on oriented textures. The top row shows the input images and the bottom row shows the corresponding interpolations created using the method.

Sweater

Towel

Grass



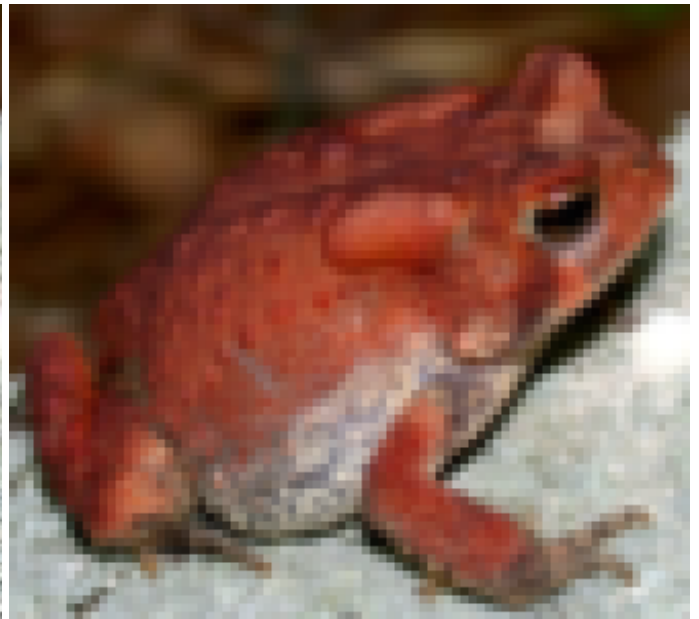


Here we compare the interpolation with several existing methods. A high resolution image is coarsened by convolving with h and the downsampling by factor 4 to create the input image. This image is then interpolated with each of the methods and compared with the original using the PSNR and MSSIM metrics. The time to compute the interpolation is also shown.

Original Image (332×300)



Input Image (83×75)



Bicubic

PSNR 24.36, MSSIM 0.6311, CPU time 0.012s

Fractal Zooming [11]

PSNR 24.50, MSSIM 0.6317

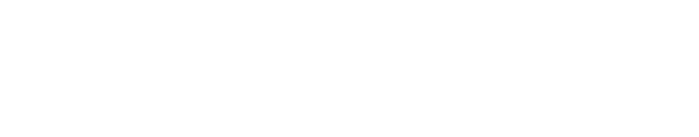


Fourier Zero-Padding with Deconvolution

PSNR 25.70, MSSIM 0.7104, CPU time 0.049s

TV Minimization [5]

PSNR 25.87, MSSIM 0.7181, CPU Time 2.72s





Contour Stencils [10]

PSNR 25.99, MSSIM 0.7256, CPU Time 0.077s



Tensor-Driven Diffusion

PSNR 26.01, MSSIM 0.7303, CPU Time 2.23s



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[image credits](#)